

# Wedge dislocation in the Geometric Theory of Defects

M. O. Katanaev \*

*Steklov Mathematical Institute,  
Gubkin St. 8, 119991, Moscow*

2 September 2002

## Abstract

We consider a wedge dislocation in the framework of elasticity theory and the geometric theory of defects. We show that the geometric theory reproduces quantitatively all the results of elasticity theory in the linear approximation. The coincidence is achieved by introducing a postulate that the vielbein satisfying the Einstein equations must also satisfy the gauge condition, which in the linear approximation leads to the elasticity equations for the displacement vector field. The gauge condition depends on the Poisson ratio, which can be experimentally measured. This indicates the existence of a privileged reference frame, which denies the relativity principle.

## 1 Introduction

Classical elasticity theory describes small deformations of elastic media without defects. Its application to media with defects is limited to individual defects because of complicated boundary conditions arising in the mathematical formulation of the problem. The field of applicability of elasticity theory is essentially limited because real solid bodies contain many defects. This raises the problem of building up the theory of defects in solid bodies. In spite of its importance and numerous attempts to solve this problem, a universal theory of defects is still lacking.

The geometric theory of defects (dislocations and disclinations) in solid bodies is a highly promising approach to this problem. This (static) model was introduced in [1], where references to earlier works can be found. The main idea of the geometric theory of defects is as follows. The infinite elastic medium without defects is represented by the Euclidean space  $\mathbb{R}^3$ , deformations being diffeomorphisms changing a metric and hence changing the extremals. The space remains flat in the sense that torsion and curvature remain zero. When dislocations are present, the infinite elastic medium is again the Euclidean space  $\mathbb{R}^3$  topologically, but the geometry changes in this case. The surface

---

\*E-mail: katanaev@mi.ras.ru

density of the Burgers vector is identified with the torsion tensor, and the curvature remains zero (a space of absolute parallelism). If the medium has a spin structure, then it may contain defects in the spin structure – disclinations. An Elastic medium with only disclinations is identified with a Riemann space, the curvature tensor being then identified with the surface density of the Frank vector. In a general case when both dislocations and disclinations are present, we have a manifold that is topologically  $\mathbb{R}^3$  but has nontrivial torsion and curvature (the Riemann–Cartan geometry).

The main independent variable in the geometric theory of defects is the vielbein, which becomes partial derivatives of the displacement vector when passing to elasticity theory. From the mathematical standpoint, this has definite advantages because the vielbein is less singular than the displacement vector in the presence of dislocations. Given a vielbein, we can construct the metric and analyze scattering of phonons on dislocations by analyzing the behavior of extremals. This problem was considered in [2, 3] and solved for an arbitrary distribution of straight parallel dislocations [4]. A similar problem in three dimensional gravity was considered in [5]. Recently, much attention was given to physical applications of the geometric theory of defects [6–13].

The geometric theory of defects describes elastic deformations, dislocations, and disclinations from a uniform standpoint. This scheme includes description of single defects as well as their continuous distribution, which can not be described in the framework of classical elasticity theory. In a certain sense, elasticity theory must then be contained in the geometric approach. General ideas of geometric approach have long been known. For example, the idea of relating dislocations to the torsion tensor was formulated in 1950s [14, 15]. Nevertheless, a quantitative agreement between the geometric approach and standard elasticity theory was not attained. The present paper fills this gap. Namely, the geometric theory of defects is shown to yield results, which in the linear approximation coincide with those of elasticity theory for a wedge dislocation. This means that classical elasticity theory is the linear approximation of the geometric theory of defects. The models agree not only qualitatively but also quantitatively.

From the mathematical standpoint, classical elasticity theory is included in the geometric theory of defects as follows. The vielbein is an independent variable in the geometric model, and it corresponds to first-order partial derivatives of the displacement vector in elasticity theory. We postulate that the vielbein must satisfy the Einstein equations for the three-dimensional metric of Euclidean signature. These equations are equilibrium equations for an elastic medium with dislocations and are second-order partial differential equations. Because the Einstein equations are covariant with respect to general coordinate transformations, we must fix the coordinate system (the gauge) to choose a unique solution. The gauge condition is an equation of no more than first order for a vielbein and corresponds to a second-order equation for a displacement vector. To achieve an agreement between the geometric theory of defects and elasticity theory, we postulate a gauge condition that coincides with the equations of classical elasticity theory in the linear approximation with respect to the displacement vector. This provides a possibility of constructing a displacement vector for a given vielbein in the chosen coordinate system, which in the linear approximation, automatically satisfies the elasticity theory equations. We thus solve the problem of quantitative agreement between the geometric theory of defects and classical elasticity theory.

We stress a basic difference between the proposed approach and the main idea of general relativity, where all coordinate systems are considered to be equivalent. In the

geometric theory of defects, we seek a solution of the Einstein equations for a vielbein in a gauge that can be reduced to the elasticity theory equations for a displacement vector. The gauge condition, as well as the elasticity theory equations, contains the experimentally observed Poisson ratio. This means that we assume the existence of a privileged coordinate system to be fixed by elasticity theory.

We consider a wedge and edge dislocations in the framework of elasticity theory in Secs. 2 and 3, and construct the displacement vector field and the corresponding metric there. In Sec. 4, we find the exact solution of the Einstein equations for a wedge dislocation in a certain gauge and show that in the linear approximation, it reproduces the results of elasticity theory not only qualitatively but also quantitatively. In Sec. 5 we compare the geometric theory of defects with the gauge approach [20, 21].

## 2 Wedge dislocation in elasticity theory

Let  $x^i$ ,  $i = 1, 2, 3$ , be Cartesian coordinates in the three dimensional Euclidean space  $\mathbb{R}^3$  with Euclidean metric  $\delta_{ij}$ . We understand a wedge dislocation to be an infinite elastic medium that coincides topologically with the Euclidean space  $\mathbb{R}^3$  with the  $z$  axis (the core of dislocation) removed and is constructed as follows. We take an infinite elastic medium without defects and cut an infinite wedge with the angle  $-2\pi\theta$ . For definiteness, we assume that the edge of the wedge coincides with the  $z$  axis as in Fig. 1. Next, the

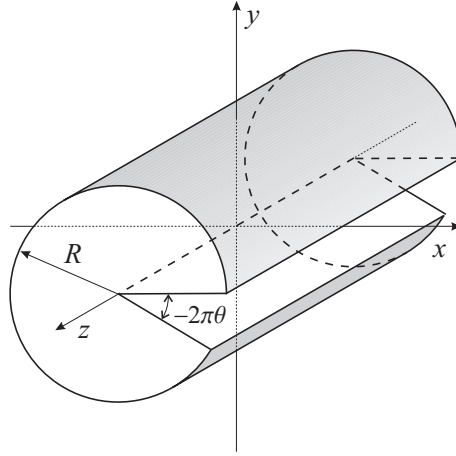


Figure 1: Wedge dislocation with the deficit angle  $2\pi\theta$ : For negative and positive  $\theta$  the wedge is respectively removed or inserted.

boundaries of the cut are symmetrically moved one to another and glued together. The medium then comes to equilibrium under the elastic forces. If the wedge is removed, then the deficit angle is assumed to be negative  $-1 < \theta < 0$ . For positive angles  $\theta$  the wedge is inserted. The initial elastic medium therefore occupies a domain larger or smaller than the Euclidean space  $\mathbb{R}^3$  depending on the sign of the deficit angle  $\theta$ . This domain is given by the following inequalities in the cylindrical coordinates  $r, \varphi, z$ :

$$0 \leq r < \infty, \quad 0 \leq \varphi \leq 2\pi\alpha, \quad -\infty < z < \infty, \quad \alpha = 1 + \theta. \quad (1)$$

We give a general definition of a dislocation for an infinite elastic medium because defects are sometimes understood differently in the scientific literature.

**Definition.** We consider an arbitrary three-dimensional manifold  $\mathbb{M}$  with a boundary  $\partial\mathbb{M}$  and a given Euclidean metric. We define a diffeomorphism  $y^i(x): \mathbb{M} \rightarrow \mathbb{R}^3 \setminus \mathbb{S}$ , where  $\mathbb{S} \subset \mathbb{R}^3$  is a submanifold of lesser dimension on which the boundary  $\partial\mathbb{M}$  is mapped. If, moreover, the displacement vector  $u^i(x) = y^i(x) - x^i$  satisfies the elasticity theory equations in the linear approximation with some boundary conditions at infinity and on the boundary  $\partial\mathbb{M}$ , then the pair  $(\mathbb{R}^3, g_{ij})$ , where

$$g_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \delta_{kl} \quad (2)$$

is the induced metric, is called a dislocation.

In the above example, the manifold  $\mathbb{M}$  is the Euclidean space  $\mathbb{R}^3$  without the wedge, its boundary  $\partial\mathbb{M}$  is the two half-planes bounding the wedge, and the submanifold  $\mathbb{S}$  is the half-plane along which the boundary of the wedge is glued. There are two points on the boundary  $\partial\mathbb{M}$  corresponding to each point of the half plane  $\mathbb{S}$  without the boundary  $\partial\mathbb{S}$ .

We make several comments on the proposed definition. If a manifold  $\mathbb{M}$  is itself the Euclidean space  $\mathbb{R}^3$  without boundary and the submanifold  $\mathbb{S}$  is empty, then we have a diffeomorphism  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ , which is just a deformation of an infinite elastic medium. Therefore, nontrivial defects arise when a manifold  $\mathbb{M}$  has a boundary  $\partial\mathbb{M}$ . In the general case, we do not require the map  $\partial\mathbb{M} \rightarrow \mathbb{S}$  to be a one-to-one correspondence, i.e., several points of a boundary  $\partial\mathbb{M}$  may be mapped to a single point of a submanifold  $\mathbb{S}$ . We stress that the definition essentially uses a fixed coordinate system in which the elasticity theory equations are written, which makes the definition noninvariant. Of course, an arbitrary coordinate system in the Euclidean space can be used, but some coordinate system must be chosen. This is essential because the elasticity theory equations contain the experimentally observed Poisson ratio (or Lamé coefficients). For definiteness, we considered the whole Euclidean space. Nevertheless, the given definition can be easily generalized to a bounded medium. For this, it suffices to replace the whole Euclidean space with a part of it bounded by some surface.

We formulate the mathematical problem for a wedge dislocation in the framework of elasticity theory. We consider the wedge dislocation as a cylinder of finite radius  $R$ . This problem has a translational symmetry along the  $z$  axis and rotational invariance in the  $x, y$  plain. We therefore use a cylindrical coordinate system. Let

$$\hat{u}_i = (\hat{u}_r, \hat{u}_\varphi, \hat{u}_z) \quad (3)$$

be coordinates of a displacement covector with respect to the orthonormal basis in a cylindrical coordinate system. This covector satisfies the equilibrium equation [16] in domain (1)

$$(1 - 2\sigma)\Delta\hat{u}_i + \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \hat{u}^j = 0, \quad (4)$$

where  $\sigma$  is the Poisson ratio,  $(-1 \leq \sigma \leq 1/2)$ ,  $\Delta$  is the Laplace operator, and  $\overset{\circ}{\nabla}_i$  is the covariant derivative for a flat Euclidean metric in the considered coordinate system.

For reference, we write explicit expressions for the divergence and Laplacian of the

displacement covector in the cylindrical coordinate system: for reference

$$\begin{aligned}\overset{\circ}{\nabla}_i \hat{u}^i &= \frac{1}{r} \partial_r (r \hat{u}^r) + \frac{1}{r} \partial_\varphi \hat{u}^\varphi + \partial_z \hat{u}^z, \\ \Delta \hat{u}_r &= \frac{1}{r} \partial_r (r \partial_r \hat{u}_r) + \frac{1}{r^2} \partial_\varphi^2 \hat{u}_r + \partial_z^2 \hat{u}_r - \frac{1}{r^2} \hat{u}_r - \frac{2}{r^2} \partial_\varphi \hat{u}_\varphi, \\ \Delta \hat{u}_\varphi &= \frac{1}{r} \partial_r (r \partial_r \hat{u}_\varphi) + \frac{1}{r^2} \partial_\varphi^2 \hat{u}_\varphi + \partial_z^2 \hat{u}_\varphi - \frac{1}{r^2} \hat{u}_\varphi + \frac{2}{r^2} \partial_\varphi \hat{u}_r, \\ \Delta \hat{u}_z &= \frac{1}{r} \partial_r (r \partial_r \hat{u}_z) + \frac{1}{r^2} \partial_\varphi^2 \hat{u}_z + \partial_z^2 \hat{u}_z.\end{aligned}$$

Proceeding from the symmetry of the problem, we seek the solution of Eq. (4) in the form

$$\hat{u}_r = u(r), \quad \hat{u}_\varphi = A(r)\varphi, \quad \hat{u}_z = 0$$

with the boundary conditions

$$\hat{u}_r|_{r=0} = 0, \quad \hat{u}_\varphi|_{r=0} = 0, \quad \hat{u}_\varphi|_{\varphi=2\pi\alpha} = -2\pi\theta r, \quad \partial_r \hat{u}_r|_{r=R} = 0. \quad (5)$$

The first three conditions are geometrical. The last condition has a simple physical meaning: the absence of external forces on the boundary of the medium. The function  $A(r)$  is found from the next to the last boundary condition in (5):

$$A(r) = -\frac{\theta}{1+\theta}r.$$

Straightforward substitution then shows that  $\varphi$  and  $z$  components of (4) are satisfied identically, and the radial component reduces to the equation

$$\partial_r (r \partial_r u) - \frac{u}{r} = D, \quad D = -\frac{1-2\sigma}{1-\sigma} \frac{\theta}{1+\theta}.$$

A general solution of this equation has the form

$$u = \frac{D}{2} r \ln r + c_1 r + \frac{c_2}{r}, \quad c_{1,2} = \text{const.}$$

The integration constant  $c_2$  is equal to zero because of the boundary condition at the origin. The constant  $c_1$  is found from the forth boundary condition in (5). As a result, we obtain the known solution for the considered problem [17]

$$\begin{aligned}\hat{u}_r &= \frac{D}{2} r \ln \frac{r}{eR}, \\ \hat{u}_\varphi &= -\frac{\theta}{1+\theta} r \varphi.\end{aligned} \quad (6)$$

We note that the radial component of the displacement vector diverges in the limit  $R \rightarrow \infty$ ; therefore, a cylinder of finite radius must be considered as a wedge dislocation.

We write the obtained solution in the Cartesian coordinate system, which we need for considering the edge dislocation,

$$\begin{aligned}u_x &= -\frac{\theta}{1+\theta} \left( \frac{1-2\sigma}{2(1-\sigma)} x \ln \frac{r}{eR} - y\varphi \right), \\ u_y &= -\frac{\theta}{1+\theta} \left( \frac{1-2\sigma}{2(1-\sigma)} y \ln \frac{r}{eR} + x\varphi \right).\end{aligned} \quad (7)$$

Linear elasticity theory is valid in the domain of small aspect ratios, which for the wedge dislocation are

$$\frac{d\hat{u}_r}{dr} = -\frac{\theta}{1+\theta} \frac{1-2\sigma}{2(1-\sigma)} \ln \frac{r}{R}, \quad \frac{1}{r} \frac{d\hat{u}_\varphi}{d\varphi} = -\frac{\theta}{1+\theta}.$$

This means that we have the right to expect correct results for the displacement field for small deficit angles  $\theta \ll 1$  and near the boundary of the dislocation ( $r \sim R$ ).

We find the metric induced by the wedge dislocation in the linear approximation with respect to the deficit angle  $\theta$ . Calculations can be performed using a general formula (2) or the known expression for the variation of the metric form

$$\delta g_{mn} = -\overset{\circ}{\nabla}_m u_n - \overset{\circ}{\nabla}_n u_m. \quad (8)$$

After simple calculations, we obtain the expression for the metric in the  $x, y$  plain:

$$ds^2 = \left(1 + \theta \frac{1-2\sigma}{1-\sigma} \ln \frac{r}{R}\right) dr^2 + r^2 \left(1 + \theta \frac{1-2\sigma}{1-\sigma} \ln \frac{r}{R} + \theta \frac{1}{1-\sigma}\right) d\varphi^2. \quad (9)$$

We compare the metric obtained as the solution of three-dimensional Einstein equations in Sec. 4 with this metric.

### 3 Edge dislocation

Wedge dislocations are rare in Nature because they require a large amount of medium to be added or removed, which results in a large expenditure of energy. Nevertheless, their analysis is of great interest because other straight dislocations can be represented as superpositions of wedge dislocations. We show this for an edge dislocation, which is one of the more widely spread dislocations. The edge dislocation with the core along the  $z$  axis is shown in Fig.2,*a*. Such a dislocation appears as the result of cutting the medium

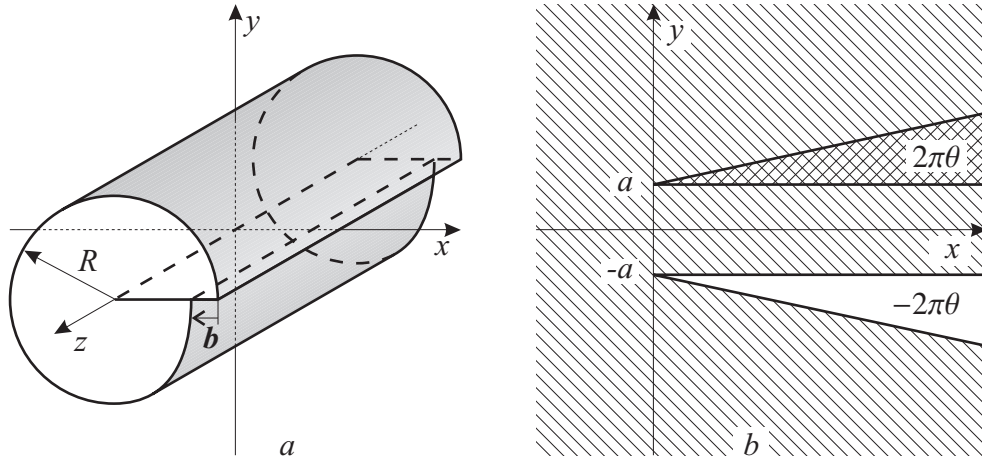


Figure 2: The edge dislocation with the Burgers vector  $\mathbf{b}$  pointing to the core of dislocation (*a*). The edge dislocation as the dipole of two wedge dislocations with negative and positive deficit angles (*b*).

along the half-plane  $y = 0$ ,  $x > 0$ , of moving the lower branch of the cut towards the  $z$  axis by a constant (far away from the core of dislocation) vector  $\mathbf{b}$  called the Burgers vector, and of subsequently gluing the branches together. To find the displacement field for an edge dislocation we can solve the corresponding boundary value problem for equilibrium equation (4) [16]. But we can proceed differently, knowing an explicit form of the displacement vector for a wedge dislocation. An edge dislocation is a dipole consisting of two wedge dislocations of positive  $2\pi\theta$  and negative  $-2\pi\theta$  deficit angles as shown in Fig.2,b. The axes of the first and the second wedge dislocations are assumed to be parallel to the  $z$  axis and intersect the  $x, y$  plane at points with the respective coordinates  $(0, a)$  and  $(0, -a)$ . The distance between axes of the wedge dislocations is equal to  $2a$ . The displacement fields for wedge dislocations far away from the origin  $r \gg a$  and up to the first order terms in  $\theta$  and  $a/r$  have the form

$$u_x^{(1)} \approx -\theta \left[ \frac{1-2\sigma}{2(1-\sigma)} x \ln \frac{r-a \sin \varphi}{eR} - (y-a) \left( \varphi - \frac{a \cos \varphi}{r} \right) \right], \quad (10)$$

$$u_y^{(1)} \approx -\theta \left[ \frac{1-2\sigma}{2(1-\sigma)} (y-a) \ln \frac{r-a \sin \varphi}{eR} + x \left( \varphi - \frac{a \cos \varphi}{r} \right) \right],$$

$$u_x^{(2)} \approx \theta \left[ \frac{1-2\sigma}{2(1-\sigma)} x \ln \frac{r+a \sin \varphi}{eR} - (y+a) \left( \varphi + \frac{a \cos \varphi}{r} \right) \right], \quad (11)$$

$$u_y^{(2)} \approx \theta \left[ \frac{1-2\sigma}{2(1-\sigma)} (y+a) \ln \frac{r+a \sin \varphi}{eR} + x \left( \varphi + \frac{a \cos \varphi}{r} \right) \right],$$

as the consequence of expression (7) for the displacement field. As far as the elasticity equations are linear, it suffices to add the displacement fields (10) and (11) to find the displacement field for the edge dislocation. Simple calculations yield the result, which is written up to a translation of the whole medium on a constant vector along the  $y$  axis

$$u_x = b \left[ \operatorname{arctg} \frac{y}{x} + \frac{1}{2(1-\sigma)} \frac{xy}{x^2 + y^2} \right], \quad (12)$$

$$u_y = -b \left[ \frac{1-2\sigma}{2(1-\sigma)} \ln \frac{r}{eR} + \frac{1}{2(1-\sigma)} \frac{x^2}{x^2 + y^2} \right],$$

and where the modulus of the Burgers vector is denoted by

$$b = |\mathbf{b}| = -2a\theta.$$

This result coincides with the expression for the displacement field obtained as the straightforward solution of the elasticity theory equations [16].

We find the metric induced by the edge dislocation. Using expression (8), we find the metric in the  $x, y$  plane in the linear approximation in  $\theta$  and  $a/r$

$$ds^2 = \left( 1 + \frac{1-2\sigma}{1-\sigma} \frac{b}{r} \sin \varphi \right) (dr^2 + r^2 d\varphi^2) - \frac{2b \cos \varphi}{1-\sigma} dr d\varphi. \quad (13)$$

## 4 Wedge dislocation in the geometric approach

We now consider the wedge dislocation from the geometric standpoint. As noted in the introduction, dislocations and disclinations in the media are respectively characterized by

nontrivial torsion and curvature. In Cartan variables, curvature and torsion tensors are

$$R_{mn}{}^{ij} = \partial_m \omega_n{}^{ij} - \omega_m{}^{ik} \omega_{nk}{}^j - (m \leftrightarrow n), \quad (14)$$

$$T_{mn}{}^i = \partial_m e_n{}^i - \omega_m{}^{ij} e_{nj} - (m \leftrightarrow n), \quad (15)$$

where  $e_m{}^i$  and  $\omega_m{}^i{}_j$  are the respective vielbein and  $\mathbb{SO}(3)$  connection. We adopt the following notations [1]. Indices  $i, j, k, l = 1, 2, 3$  refer to an orthonormal basis in the tangent space and transform under local  $\mathbb{SO}(3)$  rotations. The indices  $m, n, \dots = 1, 2, 3$  are coordinates indices, which enumerate the coordinates  $x^m$  in an arbitrary local coordinate system. If the curvature tensor vanishes (disclinations are absent), then there is a vielbein such that  $\mathbb{SO}(3)$  connection vanishes identically, at least locally,  $\omega_{mj}{}^i = 0$ . Then the whole geometry is defined by torsion tensor (15), which is unambiguously given by the vielbein  $e_m{}^i$ . The vielbein  $e_m{}^i$  uniquely defines the metric  $g_{mn} = e_m{}^i e_n{}^j \delta_{ij}$ , which satisfies the Einstein equations by assumption [1]. In spaces of absolute parallelism, we therefore find the metric by solving the Einstein equations; we then construct the vielbein, subsequently restoring the torsion tensor, which characterizes the distribution of dislocations in elastic media. We note that the same metric defines the curvature tensor at zero torsion (the Riemannian geometry). In the geometric theory of defects, the vielbein defines the torsion tensor, and we adopt this interpretation in what follows.

The geometrical action describing defects in elastic media was proposed in [1]. The equilibrium equations following from this action coincide with the three-dimensional Einstein equations for a point particle at rest and for the Euclidean signature of the metric. The time coordinate runs along the  $z$  axis. The exact solution of this problem is well known in gravity [18, 19]. It is the metric describing a conical singularity in the  $x, y$  plain, which we write in the form

$$ds^2 = \frac{1}{\alpha^2} df^2 + f^2 d\varphi^2. \quad (16)$$

Here,  $\alpha = 1 + \theta$ , where the constant  $\theta$  is proportional to the mass of a particle. We let the letter  $f$  denote the radial coordinate because we use the coordinate transformation  $f \rightarrow f(r)$  in what follows.

From the qualitative standpoint, creating a wedge dislocation is equivalent to introducing a conical singularity. Nevertheless, there is a quantitative disagreement because the metric (16) does not coincide with metric (9). This discrepancy arises because in elasticity theory, the displacement vector must satisfy the equilibrium equations after the wedge is removed and the boundaries are glued together. At the same time, after gluing for a conical singularity, the medium may be deformed arbitrarily. On the formal level, this is also manifested by induced metric (9) obtained in the framework of elasticity theory depending on the Poisson ratio, which is absent from the gravity theory. This dependence is of primary importance because the Poisson ratio can be measured experimentally. Its absence from the Einstein equations means that the geometric theory of defects must be supplied with an additional postulate.

For this, we reject the relativity principle, which is fundamental for relativity theory and equates all coordinate systems. In the geometric theory of defects, we postulate that a privileged reference frame exists and that in that coordinate system, the metric or vielbein must satisfy a condition that can be reduced to the equilibrium equations for elastic medium in the linear approximation with respect to the displacement field. The metric in the Cartesian coordinate system in the linear approximation for a displacement



field has the following form, as follows from the definition of the induced metric (2)

$$g_{mn} \approx \delta_{mn} - \partial_m u_n - \partial_n u_m.$$

Comparing this expression with equilibrium equation (4), we easily find that the gauge conditions

$$g^{mn} \overset{\circ}{\nabla}_m g_{np} + \frac{\sigma}{1-2\sigma} g^{mn} \overset{\circ}{\nabla}_p g_{mn} = 0, \quad (17)$$

$$\overset{\circ}{g}^{mn} \overset{\circ}{\nabla}_m g_{np} + \frac{\sigma}{1-2\sigma} \overset{\circ}{\nabla}_p g^T = 0 \quad (18)$$

indeed coincide with elasticity theory equations in the linear approximation. In Eq. (18), we introduce the notation  $g^T = \overset{\circ}{g}^{mn} g_{mn}$  for the trace of a metric. The testing is easiest in the Cartesian coordinate system.

Gauge conditions (17) and (18) are understood as follows. The metric  $\overset{\circ}{g}_{mn}$  is the Euclidean metric written in an arbitrary coordinate system, for example, in cylindrical or spherical coordinates. The covariant derivative  $\overset{\circ}{\nabla}_m$  is constructed for the Christoffel symbols corresponding to the metric  $\overset{\circ}{g}_{mn}$ , and hence  $\overset{\circ}{\nabla}_m \overset{\circ}{g}_{np} = 0$ . The metric  $g_{mn}$  is the metric describing a dislocation (an exact solution of the Einstein equations). The gauge conditions differ in that the contractions of indices are performed with either the dislocation metric  $g^{mn}$  or the Euclidean metric  $\overset{\circ}{g}^{mn}$  in the respective first and second cases, and this does not alter the linear approximation. If a solution of the Einstein equations satisfies one of the conditions (17)–(18) written, for example, in cylindrical coordinates, then we shall say that a solution is found in cylindrical coordinate system. We can seek an analogous solution in the Cartesian, spherical or any other coordinate system.

The gauge condition can be also written for a vielbein  $e_m^i$  defined by the equation

$$g_{mn} = e_m^i e_n^j \delta_{ij}.$$

We must bare in mind that the vielbein is defined by the metric up to local  $\mathbb{SO}(3)$  rotations acting on the indices  $i, j$ . Therefore, it may have different linear approximations. We consider two possibilities (in the Cartesian coordinate system),

$$e_{mi} \approx \delta_{mi} - \partial_m u_i, \quad (19)$$

$$e_{mi} \approx \delta_{mi} - \frac{1}{2}(\partial_m u_i + \partial_i u_m), \quad (20)$$

where the index is lowered using the Kronecker symbol. Two gauge conditions on the vielbein correspond to these possibilities and to condition (18)

$$\overset{\circ}{g}^{mn} \overset{\circ}{\nabla}_m e_{ni} + \frac{1}{1-2\sigma} \overset{\circ}{e}^m_i \overset{\circ}{\nabla}_m e^T = 0, \quad (21)$$

$$\overset{\circ}{g}^{mn} \overset{\circ}{\nabla}_m e_{ni} + \frac{\sigma}{1-2\sigma} \overset{\circ}{e}^m_i \overset{\circ}{\nabla}_m e^T = 0, \quad (22)$$

where  $e^T = \overset{\circ}{e}^m_i e_m^i$ . These conditions differ by the coefficients in front of the second term. We note that in a curvilinear coordinate system, the flat  $\mathbb{SO}(3)$  connection acting on the indices  $i$  and  $j$  must be added to the covariant derivative  $\overset{\circ}{\nabla}_m$ . Other gauge conditions

having the same linear approximation can also be written. The question of the correct choice is beyond the scope of this paper. At this stage, we only want to show that the gauge condition must be imposed and that it depends on the Poisson ratio, which is an experimentally observed quantity.

Gauge conditions (21) and (22) are themselves first-order differential equations and admit some arbitrariness. To fix a solution uniquely, we must therefore impose boundary conditions on the vielbein in any particular problem.

The problem of describing dislocations in the framework of the geometric theory of defects thus reduces to solving the Einstein equations with a gauge condition on the vielbein. For brevity, we call the gauge condition (on a metric or on a vielbein) that reduces to the elasticity theory equations for the displacement field in the linear approximation the *elastic gauge*. We impose elastic gauge (22) as the simplest gauge in the case of a wedge dislocation. The problem can be solved in two ways. First, the gauge condition can be directly inserted in the Einstein equations. Second, we can seek the solution in any suitable coordinate system and then find the coordinate transformation providing satisfaction of the gauge condition.

Because exact solution (16) for the metric is known, we follow the simpler, second way. The vielbein corresponding to metric (16) can be chosen in the form

$$e_r^{\hat{r}} = \frac{1}{\alpha}, \quad e_\varphi^{\hat{\varphi}} = f.$$

Here, the hat symbol over an index means that it refers to the orthonormal coordinate system, and an index without a hat is a coordinate index. Components of the vielbein are the square roots of the respective metric components and hence admit symmetric linear approximation (20). Because the wedge dislocation is symmetric with respect to rotations in the  $x, y$  plain, we perform the transformation of the radial coordinate  $f \rightarrow f(r)$ , after which the transformed vielbein components become

$$e_r^{\hat{r}} = \frac{f'}{\alpha}, \quad e_\varphi^{\hat{\varphi}} = f, \quad (23)$$

where the prime denotes differentiation with respect to  $r$ . The vielbein corresponding to the Euclidean metric can be chosen in the form

$$\overset{\circ}{e}_r^{\hat{r}} = 1, \quad \overset{\circ}{e}_\varphi^{\hat{\varphi}} = r. \quad (24)$$

The Christoffel symbols  $\overset{\circ}{\Gamma}_{mn}^p$  and  $\mathbb{O}(3)$  connection  $\overset{\circ}{\omega}_{mi}^j$  defining the covariant derivative correspond to this vielbein. We write only nontrivial components:

$$\begin{aligned} \overset{\circ}{\Gamma}_{r\varphi}^\varphi &= \overset{\circ}{\Gamma}_{\varphi r}^\varphi = \frac{1}{r}, & \overset{\circ}{\Gamma}_{\varphi\varphi}^r &= -r, \\ \overset{\circ}{\omega}_{\varphi\hat{r}}^{\hat{\varphi}} &= -\overset{\circ}{\omega}_{\varphi\hat{\varphi}}^{\hat{r}} = 1. \end{aligned}$$

Straightforward substitution of the vielbein in the gauge condition (22) yields the Euler differential equation for the transition function:

$$\frac{f''}{\alpha} + \frac{f'}{\alpha r} - \frac{f}{r^2} + \frac{\sigma}{1-2\sigma} \left( \frac{f''}{\alpha} + \frac{f'}{r} - \frac{f}{r^2} \right) = 0.$$

Its general solution depends on two constants  $C_{1,2}$ ,

$$f = C_1 r^{n_1} + C_2 r^{n_2},$$

where the exponents  $n_{1,2}$  are the roots of the quadratic equation

$$n^2 + n \frac{\sigma\theta}{1-\sigma} - \alpha = 0.$$

To fix the constants of integration, we impose the boundary conditions on the vielbein:

$$e_r^{\hat{r}}|_{r=R} = 1, \quad e_\varphi^{\hat{\varphi}}|_{r=0} = 0. \quad (25)$$

The first boundary condition corresponds to the fourth boundary condition on the displacement vector (5), and the second condition means the absence of the angular component of the deformation tensor at the core of dislocation. These requirements define the integration constants as

$$C_1 = \frac{\alpha}{n_1 R^{n_1-1}}, \quad C_2 = 0.$$

The metric corresponding to the obtained vielbein is

$$ds^2 = \left(\frac{r}{R}\right)^{2(n_1-1)} \left(dr^2 + \frac{\alpha^2 r^2}{n_1^2} d\varphi^2\right), \quad (26)$$

where

$$n_1 = \frac{-\theta\sigma + \sqrt{\theta^2\sigma^2 + 4(1+\theta)(1-\sigma)^2}}{2(1-\sigma)}.$$

This is the solution of the problem.

In the linear approximation in  $\theta$ , we have

$$n_1 \approx 1 + \theta \frac{1-2\sigma}{2(1-\sigma)},$$

and it is easy to show that metric (26) indeed coincides with metric (9) obtained in the framework of elasticity theory. The essential difference, however, appears beyond the perturbation theory. Metric (9) is singular at the origin, whereas metric (26) obtained beyond the perturbative expansion is regular.

The problem of reconstructing the displacement field for a given metric can be reduced to solving Eqs. (2), where metric (26) must be inserted in the right-hand side with boundary conditions (5). We do not consider this problem here.

## 5 Comparison with the gauge approach

We compare the geometric theory of defects with the gauge theory of dislocations and disclinations considered in [20, 21]. In the gauge approach, the vielbein is

$$e_m^i = \partial_m y^i + y^j \omega_{mj}^i + \phi_m^i, \quad (27)$$

where  $y^i(x)$  is a section of a principle fibre bundle with the three dimensional translation structure group  $\mathbb{T}(3)$ , and  $\phi_m^i(x)$  is the gauge field for the subgroup of translations.

This construction is analogous to the gauge theory for the Poincaré group considered, for example, in [22]. Under local rotations  $S_j^i(x) \in \mathbb{SO}(3)$  and translations  $a^i(x) \in \mathbb{T}(3)$  the fields are transformed according to the rules

$$\begin{aligned} y^i &= y^j S_j^i + a^i, \\ \omega_{mj}^i &= S_j^{-1k} \omega_{mk}^l S_l^i - S_j^{-1k} \partial_m S_k^i, \\ \phi_m^i &= \phi_m^j S_j^i - a^j (S_j^{-1k} \omega_{mk}^l S_l^i - S_j^{-1k} \partial_m S_k^i) - \partial_m a^i. \end{aligned}$$

It is easy to show that the transformation law for the vielbein is

$$e_m^i = e_m^j S_j^i.$$

This means that the vielbein is invariant under translations.

In the gauge approach [20, 21] the fields  $y^i(x)$  and  $\phi_m^i(x)$  are treated as independent variables. In the geometric approach [1], vielbein (27) is the only independent variable. The following consideration justifies the second choice. The Lagrangian invariant with respect to local translations can depend on the fields  $y^i$  and  $\phi_m^i$  only through invariant combination (27). This follows because the semidirect product of the rotational group  $\mathbb{SO}(3)$  on the group of translations  $\mathbb{T}(3)$  is not semisimple and does not admit bi-invariant nondegenerate metric. There exists a gauge in which the vielbein coincides with the gauge field of translations,  $e_m^i = \phi_m^i$ , because we can always choose the parameter of translations such that  $y^i(x) = 0$  is insured. As a result, we obtain the Riemann–Cartan geometry, which is precisely the starting point of geometric approach [1]. The resulting model is invariant under general coordinate transformations and local rotations, but the invariance under local translations is lost.

## 6 Conclusion

We have described a wedge dislocation in two ways: in the framework of classical elasticity theory and in the geometric approach. For the first time, the geometric theory of defects is shown to reproduce quantitatively all the results obtained in elasticity theory in the linear approximation. This is worth mentioning because in solving the problem, an exact solution of the nonlinear Einstein equations with complicated gauge conditions was found. The equations of the geometric theory of defects are complicated, but they simultaneously allow an ample opportunity for solving those problems that seem insuperable in the elasticity theory.

We concentrate on the merits of the geometric approach demonstrated in the present paper. From the physical standpoint, a wedge dislocation has rotational symmetry because properties of medium are independent of the place where the wedge is removed or inserted. The corresponding displacement field (the main independent variable in the elasticity theory) does not admit the rotational symmetry and has a discontinuity on the gluing surface. Simultaneously, the gluing surface is not distinguished physically, and only the core of dislocation is essential. Either the vielbein or the corresponding metric are independent variables in the geometric theory of defects. Being a solution of the Einstein equations, the vielbein in the geometric theory of defects has rotational symmetry and is regular everywhere except the axis, where the metric has a conical singularity. Hence, the geometric variables seem to be more natural. From the mathematical standpoint, the

boundary conditions on the displacement field become so complicated for several dislocations that solving the elasticity theory equations seems impossible. In the geometric theory of defects, the presence of several dislocations results only in the modifying the right-hand side of Einstein equations. The problem is easily generalized to the important case of a continuous distribution of defects, where the right-hand side becomes smooth. For a continuous distribution of defects, the displacement field does not exist, but the vielbein field can be defined, and this is important.

We consider gauge conditions (18), (21), and (22). Being written in terms of the displacement vector, they yield the equations of nonlinear elasticity theory [16]. From this standpoint, the geometric theory of defects yields the solution of the problem of nonlinear elasticity theory. Moreover, metric (26) is then an exact solution of the problem and is regular on the whole space except the core of dislocation, where it has a conical singularity. For comparison, we note that metric (13) obtained in the elasticity theory can be used only for small deficit angles and near the boundary of dislocation.

At first glance, the possibility of expressing the vielbein satisfying the Einstein equations through the displacement field satisfying the elasticity theory equations seems surprising. The mathematical reason for this is simple. Any solution of the Einstein equations is defined up to a diffeomorphism. Because the displacement vector field parameterizes diffeomorphisms, we have the freedom to require that it satisfies the elasticity theory equations.

In the present paper, we have shown that the elasticity theory can be imbedded in the geometric theory of defects by imposing a gauge condition on the vielbein such that it reproduces the elasticity theory equations for the displacement vector. Because the gauge condition depends explicitly on an experimentally observed constant (the Poisson ratio) we reject the relativity principle. In other words, there exists a distinguished coordinate system. If the geometric theory of defects is inverted, and gravity theory is considered as the theory of the elastic ether with defects, then a field of speculations arise. For example, the problem of measuring the Poisson ratio of ether can be posed. Only future investigations can answer to such questions.

In spite of the manifest merits of the geometric approach, many problems remain open. In particular, there are many gauge conditions reducing to the elasticity theory equations in the linear approximation, and it is not clear what condition is to be chosen. The investigation of how to solve the Einstein equations directly in the elastic gauge remains. These and other related questions are out of the scope of this paper.

The author is very grateful to I. V. Volovich for the discussion of the paper. The work is supported by the Russian Foundation for Basic Research (Grant Nos. 96-15-96131 and 02-01-01084).

## References

- [1] M. O. Katanaev and I. V. Volovich. Theory of defects in solids and three-dimensional gravity. *Ann. Phys.*, 216(1):1–28, 1992.
- [2] F. Moraes. Geodesics around a dislocation. *Phys. Lett.*, A214(3,4):189–192, 1996.
- [3] A. de Padua, F. Parisio-Filho, and F. Moraes. Geodesics around line defects in elastic solids. *Phys. Lett. A*, 238:153–158, 1998.

- [4] M. O. Katanaev and I. V. Volovich. Scattering on dislocations and cosmic strings in the geometric theory of defects. *Ann. Phys.*, 271:203–232, 1999.
- [5] G. Clement. Flat wormholes from straight cosmic strings. *J. Math. Phys. B*, 38:5807–5819, 1997.
- [6] S. Azevedo, C. Furtado, and F. Moraes. Charge localization around disclinations in monolayer graphite. *phys. stat. sol. (b)*, 207:387–392, 1998.
- [7] A. P. Balachandran, V. John, A. Momen, and F. Moraes. Anomalous defects and their quantized transverse conductivities. *Int. J. Mod. Phys. A*, 13(5):841–861, 1998.
- [8] R. Bausch, R. Schmitz, and L. A. Turski. Single-particle quantum states in a crystal with topological defects. *Phys. Rev. Lett.*, 80(11):2257–2260, 1998.
- [9] S. Azevedo and F. Moraes. Topological aharonov–bohm effect around a disclination. *Phys. Lett. A*, 246:374–376, 1998.
- [10] A. De S. Barbosa, E. R. Bezerra De Mello, and V. B. Bezerra. Coulomb interaction between two charged particles on a cone. *Int. J. Mod. Phys. A*, 14(22):3565–3580, 1999.
- [11] C. Furtado and F. Moraes. Landau levels in the presence of a screw dislocation. *Europhys. Lett.*, 45(3):279–282, 1999.
- [12] S. Azevedo and J. Pereira. Double aharonov–bohm effect in a medium with a disclination. *Phys. Lett. A*, 275:463–466, 2000.
- [13] M. Lazar. An elastoplastic theory of dislocations as a physical field theory with torsion. *J. Phys. A*, 35:1983–2004, 2002.
- [14] K. Kondo. On the geometrical and physical foundations of the theory of yielding. In *Proc. 2nd Japan Nat. Congr. Applied Mechanics*, page 41, Tokyo, 1952.
- [15] B. A. Bilby, R. Bullough, and E. Smith. *Proc. Roy. Soc. London*, A231:263, 1955.
- [16] L. D. Landau and E. M. Lifshits. *Theory of Elasticity*. Pergamon, Oxford, 1970.
- [17] A. M. Kosevich. *Physical Mechanics of Real Crystals*. Naukova Dumka, Kiev, 1981 [in Russian].
- [18] A. Staruszkiewicz. Gravitational theory in three-dimensional space. *Acta Phys. Polon.*, 24(6(12)):735–740, 1963.
- [19] G. Clement. Field-theoretic particles in two space dimensions. *Nucl. Phys. B*, 114:437–448, 1976.
- [20] A. Kadić and D. G. B. Edelen. A gauge theory of dislocations and disclinations, Springer–Verlag, Berlin – Heidelberg, 1983.
- [21] C. Malyshev. The  $T(3)$ -gauge model, the Einstein-like gauge equation, and Volterra dislocations with modified asymptotics. *Ann. Phys.*, 286:249–277, 2000.
- [22] M. O. Katanaev. Gauge theory for the Poincaré group. *Theor. Math. Phys.* 54:248–252, 1983.